A No-Go Theorem for Deterministic Born Recovery from Branch-Conditioned Records

Claudio Irrgang

March 17, 2026

Abstract

Under pointer-commuting dynamics ($[H, P_k] = 0$), the branch-conditioned record laws are amplitude-independent. We consider deterministic measurable selectors whose input is restricted to such records, under assumptions (A1)–(A5). Any such selector has an output distribution of the form $\mu_F = C\, w(\mathbf{a})$, where C is a fixed column-stochastic matrix and $w(\mathbf{a}) = (|a_k|^2)$ are the Born weights. Reproducing the Born rule on any set whose Born-weight image contains a nonempty open subset of $\mathrm{int}(\Delta_{d_S-1})$ requires $C = I$; this in turn holds if and only if the conditional record laws ν_k admit a measurable partition $\{D_k\}$ with $\nu_k(D_k) = 1$ for each k. This is a scoped obstruction result for the selector and dynamics class defined by (A1)–(A5), not a general no-go theorem for quantum measurement. The result does not exclude theories in which amplitude information reaches the selector through channels outside this framework.

Contents

1 Introduction **3**

2 Setup and Main Theorem **4**
2.1 Hilbert space and premeasurement 4
2.2 Assumptions 5
2.3 Sector decomposition 6
2.4 Sector-state amplitude independence 7
2.5 Column-stochastic rigidity 7
2.6 The no-go theorem 8

3 Supporting Closures **10**
3.1 Frozen diagonal 10
3.2 Robustness under perturbation 11
3.3 General Lipschitz bound 12

4 Deterministic and Stochastic Trajectories **13**
4.1 Frozen branch weights 14
4.2 Stochastic trajectory model 14
4.3 Comparison 15
4.4 Escape channels 15
4.5 Scope of the comparison 16

5 Discussion **16**
5.1 What the no-go result establishes 16
5.2 When is Born recovery achievable? 16
5.3 Scope of the theorem 18
5.4 Decomposition as contribution 18
5.5 Relation to decoherence and trajectory models 19
5.6 Relation to existing no-go theorems 19
5.7 Limits and future directions 19

6 Conclusion **20**

A Assumption Dependencies **21**

B Auxiliary Proof Details **22**

C Symbolic Verification **23**

1 Introduction

The quantum measurement problem, in its modern formulation, asks how definite outcomes arise from unitary evolution of entangled states [10, 11]. Decoherence theory has clarified a substantial part of this question: interaction with an environment suppresses interference between branches [13], selects a preferred pointer basis via einselection [14, 15], and produces stable records in the apparatus and environment [8, 16]. What decoherence does not provide, by itself, is a mechanism for selecting one outcome with the correct Born-rule probabilities.

This paper investigates a precise version of that gap. We consider a standard three-stage decomposition of measurement: premeasurement (which correlates system eigenstates with apparatus records), decoherence (which stabilizes those records via environment coupling), and outcome selection (which produces a definite result). Within this decomposition, we ask: can the third stage be implemented by a deterministic rule operating on the records produced by the second?

The pointer-commuting family

We study post-premeasurement models whose Hamiltonians commute with the pointer projectors, $[H, P_k] = 0$, so that the branch structure established by premeasurement is preserved by the subsequent dynamics. Any system-decoupled Hamiltonian $H = I_S \otimes H_{AE}$ is pointer-commuting, as is any system-coupled Hamiltonian $H = \sum_k |v_k\rangle\langle v_k| \otimes h_k$ whose coupling respects the pointer basis. Some standard decoherence models with pointer-diagonal couplings [8, 16] fall within this class.

The outcome selector

An *outcome selector* is a deterministic function $F\colon \mathbf{D} \to \{1, \ldots, d_S\}$ that maps a measurable record space $\mathbf{D}$ of branch-conditioned apparatus–environment record data to an outcome label. The selector's input is restricted to measurable branch-conditioned record variables in the common record space $\mathbf{D}$; it does not have access to the branch label k or to the system amplitude vector $\mathbf{a}$. This domain restriction is codified in (A5).

Thesis

Within the present model class, branch-conditioned apparatus–environment records have laws that are independent of the system amplitudes. Any deterministic measurable selector restricted to these records yields an output distribution $\mu_F = C\, w(\mathbf{a})$ for a fixed column-stochastic matrix C. Within (A1)–(A5), Born-rule agreement on any set whose Born-weight image contains a nonempty open subset of $\mathrm{int}(\Delta_{d_S-1})$ forces $C = I$. By Theorem 2.7(iv), this holds if and only if the conditional record laws ν_k admit a measurable partition $\{D_k\}$ with $\nu_k(D_k) = 1$ for each k.

Contributions

The paper provides a typed decomposition of the obstruction and a structural localization of where deterministic Born recovery fails within this framework:

1. A no-go theorem (Theorem 2.7): amplitude-independence of branch-conditioned records (Lemma 2.3) forces the output to be a fixed column-stochastic transform of the Born weights.

2. A general Lipschitz bound (Proposition 3.7): within the present finite-dimensional framework, the Born mismatch persistence argument extends to perturbations $H_\delta = H_0 + \delta V$ with $[H_0, P_k] = 0$, with an explicit dimension-independent bound.

3. A classification of the partition condition (Theorem 5.1): in this setting, Born recovery is structurally stable precisely under pointer-commuting.

4. A structural comparison (Section 4) with a canonical SSE benchmark, exhibiting state-dependent stochastic variability absent from the deterministic regime.

Supporting results include: form-invariance of the sector decomposition in any compatible pointer basis, with the full no-go requiring the corresponding primed premeasurement and selector hypotheses (Theorem 3.3); and a frozen-diagonal result showing that pointer-basis occupation probabilities are constant under this evolution (Corollary 3.1).

Beyond the no-go itself, Theorem 2.7(iv) identifies, within the selector and dynamics class of (A1)–(A5), the condition under which deterministic Born recovery is possible. Theorem 2.7(iv) shows that this occurs precisely when the branch-conditioned record laws admit a measurable partition that identifies the branch almost surely. Within (A1)–(A5), this reformulates the problem as one of distinguishability of the branch-conditioned record laws generated by the sector dynamics. A natural extension is to characterize which post-premeasurement Hamiltonians and record maps in this framework yield branch-conditioned record laws with that almost-sure identification property.

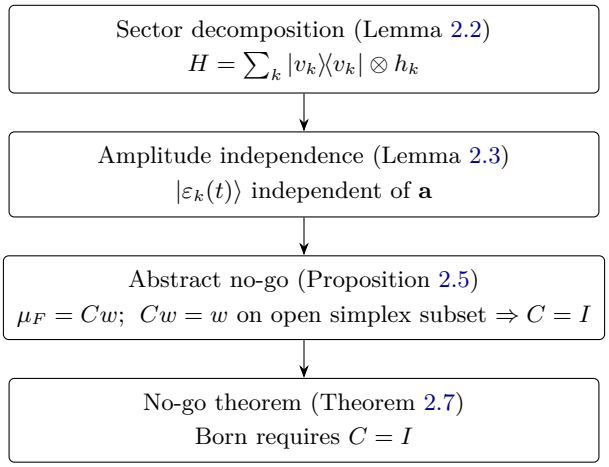

Figure 1: Proof spine of the main theorem. Each box states the result and its key content; edge labels indicate assumptions consumed.

The no-go applies within the dynamical and selector class defined by (A1)–(A5); it does not exclude theories that supply amplitude information through other channels and does not claim to resolve the measurement problem.

2 Setup and Main Theorem

2.1 Hilbert space and premeasurement

We work in a tripartite Hilbert space

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_E, \tag{1}$$

where $\mathcal{H}_S$ is the system ($\dim \mathcal{H}_S = d_S < \infty$), $\mathcal{H}_A$ the apparatus ($\dim \mathcal{H}_A \geq d_S$), and $\mathcal{H}_E$ the environment, all finite-dimensional. We write I_{AE} for the identity operator on $\mathcal{H}_A \otimes \mathcal{H}_E$. A distinguished orthonormal basis $\{|v_k\rangle\}_{k=1}^{d_S}$ of $\mathcal{H}_S$ is the measurement basis singled out by the premeasurement coupling, with associated projectors

$$P_k = |v_k\rangle\langle v_k| \otimes I_{AE}. \tag{2}$$

Since $\{|v_k\rangle\}$ is an orthonormal basis, $\sum_k P_k = I_{\mathcal{H}}$. The P_k are pairwise orthogonal sector projectors.

A premeasurement unitary U_{SA} correlates system eigenstates with apparatus records:

$$U_{SA}\colon |v_k\rangle|0\rangle_A \longmapsto |v_k\rangle|k\rangle_A, \tag{3}$$

where $|0\rangle_A$ is the apparatus ready state and $\{|k\rangle_A\}$ are orthonormal apparatus pointer states. This map is isometric on $\mathrm{span}\{|v_k\rangle|0\rangle_A\}$ and therefore extends to a unitary on $\mathcal{H}_S \otimes \mathcal{H}_A$; the choice of extension does not affect (4). For an initial system state $|\psi\rangle = \sum_k a_k|v_k\rangle$ and environment state $|\lambda\rangle_E$, premeasurement produces

$$|\Psi_0\rangle = \sum_{k=1}^{d_S} a_k\,|v_k\rangle|k\rangle_A|\lambda\rangle_E, \qquad \sum_k |a_k|^2 = 1. \tag{4}$$

Normalization follows from $\langle\psi|\psi\rangle = 1$, orthonormality of $\{|v_k\rangle\}$ and $\{|k\rangle_A\}$, and normalization of $|\lambda\rangle_E$.

2.2 Assumptions

The main theorem rests on five assumptions, referenced by label throughout.

(A1) Finite Hilbert space. $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_E$ with all three factors finite-dimensional, $\dim \mathcal{H}_S = d_S < \infty$, and $\dim \mathcal{H}_A \geq d_S$.

(A2) Pointer basis. A distinguished orthonormal basis $\{|v_k\rangle\}_{k=1}^{d_S}$ of $\mathcal{H}_S$ with associated projectors $P_k = |v_k\rangle\langle v_k| \otimes I_{AE}$. This is the basis correlated with orthonormal apparatus record states by the premeasurement map U_{SA}.

(A3) Pointer-commuting dynamics. The post-premeasurement Hamiltonian satisfies $[H, P_k] = 0$ for all k. Equivalently, the post-premeasurement dynamics preserves the sectors $P_k\mathcal{H}$.

(A4) Perfect premeasurement. There exists a ready state $|0\rangle_A$ such that $U_{SA}\colon |v_k\rangle|0\rangle_A \to |v_k\rangle|k\rangle_A$. Since the full premeasurement operator is $U_{SA} \otimes I_E$, the post-premeasurement state is $|\Psi_0\rangle = \sum_k a_k|v_k\rangle|k\rangle_A|\lambda\rangle_E$ as in (4).

(A5) Deterministic selector. Let $(\mathbf{D}, \mathcal{F}_{\mathbf{D}})$ be a measurable record space. The selector input consists of branch-conditioned apparatus–environment record variables and fixed amplitude-independent parameters; the selector is any measurable functional of these inputs. Branch-specific standard Borel record spaces are embedded into a common measurable space $\mathbf{D}$. The selector is a measurable map $F\colon \mathbf{D} \to \{1, \ldots, d_S\}$ (codomain equipped with the discrete σ-algebra). Neither the branch label k nor the amplitude vector (a_k) may be supplied as auxiliary input to F.

Remark 2.1 (Non-circularity of (A5)). (A5) restricts the selector's admissible input class to record data on $\mathcal{H}_A \otimes \mathcal{H}_E$; it does not assume amplitude-independence. Amplitude-independence of the record laws is a consequence of (A1)–(A4), proved in Lemma 2.3.

Assumptions (A1)–(A4) define the physical setup. Assumption (A5) specifies the selector class under consideration. The theorem does not assume that the branch-conditioned laws ν_k are identical; only that they are amplitude-independent.

Notation. Throughout, total variation distance is the $\frac{1}{2}\ell^1$ norm: $\|\mu - \nu\|_{\mathrm{TV}} \coloneqq \frac{1}{2}\sum_j |\mu_j - \nu_j| = \sup_A |\mu(A) - \nu(A)|$.
The main theorem chain is:

1. Lemma 2.2: sector decomposition from $[H, P_k] = 0$.

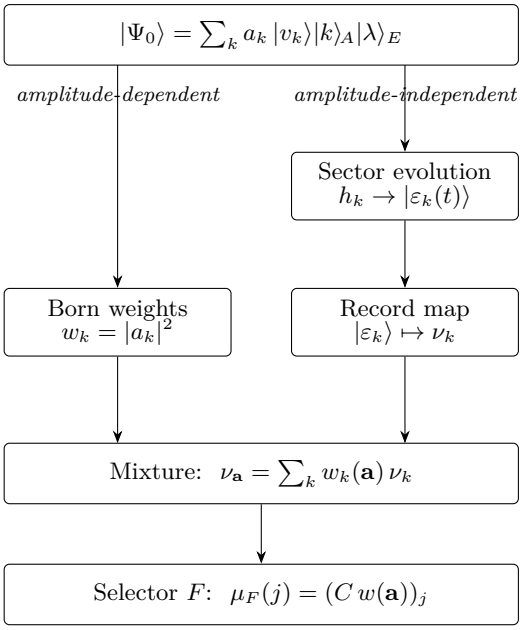

Figure 2: Data flow in the pointer-commuting framework. The right lane (sector evolution, record map) is amplitude-independent (Lemma 2.3). The left lane (Born weights) carries all amplitude dependence. These combine in the mixture; the selector output depends on amplitudes only through the Born weights.

2. Lemma 2.3: sector-evolved apparatus–environment states and their record laws are amplitude-independent.

3. Theorem 2.7: deterministic selectors on such records cannot reproduce the Born distribution on any set whose Born-weight image contains a nonempty open subset of the simplex interior.

2.3 Sector decomposition

Lemma 2.2 (Sector decomposition)**.** *Under* (A1)–(A3)*, the pointer-commuting condition* $[H, P_k] = 0$ *for all* k *implies*

$$H = \sum_{k=1}^{d_S} |v_k\rangle\langle v_k| \otimes h_k, \tag{5}$$

where each h_k *is a self-adjoint operator on* $\mathcal{H}_A \otimes \mathcal{H}_E$ *determined entirely by* H *and the pointer basis.*

Proof. Write $H = \sum_{k,\ell} |v_k\rangle\langle v_\ell| \otimes H_{k\ell}$ with $H_{k\ell}$ operators on $\mathcal{H}_A \otimes \mathcal{H}_E$. Then $[H, P_j] = 0$ gives

$$\sum_k |v_k\rangle\langle v_j| \otimes H_{kj} \;-\; \sum_\ell |v_j\rangle\langle v_\ell| \otimes H_{j\ell} \;=\; 0.$$

For $k \neq j$, the operator $|v_k\rangle\langle v_j|$ appears only in the first sum, with coefficient H_{kj}. Since the operators $\{|v_k\rangle\langle v_\ell|\}$ are linearly independent, $H_{kj} = 0$ for all $k \neq j$. Thus all off-diagonal blocks vanish, and setting $h_k := H_{kk}$ yields (5). Self-adjointness of H forces each h_k to be self-adjoint. □

The sector decomposition means that each pointer sector evolves independently under its own Hamiltonian h_k. Applying (5) to (4) yields

$$|\Psi(t)\rangle = \sum_{k=1}^{d_S} a_k\,|v_k\rangle \otimes e^{-ih_k t}\big(|k\rangle_A|\lambda\rangle_E\big). \tag{6}$$

2.4 Sector-state amplitude independence

Lemma 2.3 (Sector-state amplitude independence)**.** *Under* (A1)–(A4)*, for each k define the sector-evolved apparatus–environment state by*

$$|\varepsilon_k(t)\rangle := e^{-ih_k t}(|k\rangle_A|\lambda\rangle_E). \tag{7}$$

The state $|\varepsilon_k(t)\rangle$ is independent of the amplitude vector $\mathbf{a} = (a_1, \ldots, a_{d_S})$. When $a_k \neq 0$, the normalized branch-conditioned state is

$$\frac{P_k|\Psi(t)\rangle}{\|P_k|\Psi(t)\rangle\|} = \frac{a_k}{|a_k|}\,|v_k\rangle \otimes |\varepsilon_k(t)\rangle \qquad (a_k \neq 0). \tag{8}$$

The apparatus–environment factor $|\varepsilon_k(t)\rangle$ is amplitude-independent; the only dependence on $\mathbf{a}$ is the scalar phase factor $a_k/|a_k|$, which carries no information in the AE factor.

Proof. By Lemma 2.2, $P_k|\Psi(t)\rangle = a_k|v_k\rangle \otimes e^{-ih_k t}(|k\rangle_A|\lambda\rangle_E)$. The operator h_k depends only on H and the pointer basis, and the input $|k\rangle_A|\lambda\rangle_E$ is amplitude-independent. Hence $|\varepsilon_k(t)\rangle = e^{-ih_k t}(|k\rangle_A|\lambda\rangle_E)$ is independent of $\mathbf{a}$. Because the record map acts only on apparatus–environment variables, the scalar phase $a_k/|a_k|$ in (8) does not enter the induced record. Since the record map is fixed and amplitude-independent, the resulting apparatus–environment record law ν_k is independent of $\mathbf{a}$. □

Remark 2.4. The full superposed state $|\Psi(t)\rangle$ retains the amplitudes as coefficients. The claim is about the *per-sector* evolved state, not the unreduced global state. The Born weights $|a_k|^2$ remain in the global state but not in the sector-evolved state, nor in any record law derived from it via an amplitude-independent record map (i.e., inaccessible within the selector class of (A5); theories that supply amplitude information through other channels are not excluded). Corollary 3.1 makes this precise: the Born weights persist in the reduced system state $\rho_S(t)$, but under (A5) the selector accesses only AE record variables, not $\rho_S(t)$.

Since $|\varepsilon_k(t)\rangle$ is amplitude-independent (Lemma 2.3), any record variable $R_k := r(|\varepsilon_k(t)\rangle, \theta)$ defined by a fixed amplitude-independent measurable record map r (with fixed amplitude-independent parameters θ) has a law ν_k independent of $\mathbf{a}$; the same holds for any measurable pushforward of ν_k. This is used in Theorem 2.7(i).

Object	**Depends on a?**	**Reference**
Sector Hamiltonian h_k	No	Lemma 2.2
Sector AE state $\|\varepsilon_k(t)\rangle$	No	Lemma 2.3
Conditional record law ν_k	No	Lemma 2.3 + record map
Born weight $w_k(\mathbf{a}) = \|a_k\|^2$	**Yes**	Definition
Unconditional law $\nu_{\mathbf{a}} = \sum_k w_k \nu_k$	**Yes**, via $w(\mathbf{a})$	Proposition 2.5
Selector output $\mu_F = C\,w(\mathbf{a})$	**Yes**, via $w(\mathbf{a})$	Theorem 2.7

Figure 3: Amplitude dependence of each object in the framework. The top three rows are amplitude-independent; amplitude information enters only through the Born weights and then propagates through the mixture and selector.

2.5 Column-stochastic rigidity

The following proposition is measure-theoretic: it uses no Hilbert-space structure, only measurability and sector weights.

Proposition 2.5 (Abstract no-go)**.** *Let* $\mathbf{D}$ *be a measurable record space. For each sector* $k \in \{1, \ldots, n\}$*, let the conditional record law* ν_k *on* $\mathbf{D}$ *be independent of a parameter* $\mathbf{a} \in \mathcal{A}$*, and let* $w_k(\mathbf{a}) \geq 0$ *with* $\sum_k w_k = 1$ *be sector weights depending on* $\mathbf{a}$*. Assume the unconditional record law at parameter* $\mathbf{a}$ *is the mixture*

$$\nu_{\mathbf{a}} = \sum_{k=1}^{n} w_k(\mathbf{a})\, \nu_k. \tag{9}$$

Let $F \colon \mathbf{D} \to \{1, \ldots, n\}$ *be any measurable selector, and define* $c_{jk} \coloneqq \nu_k[F = j]$ *and* $\mu_F(j) \coloneqq \nu_{\mathbf{a}}(F = j)$*. Then:*

(i) *Each* c_{jk} *is a fixed constant (independent of* $\mathbf{a}$*), and the matrix* $C = (c_{jk})$ *is column-stochastic:* $c_{jk} \geq 0$ *and* $\sum_j c_{jk} = 1$ *for each* k*.*

(ii) *The overall output law is*

$$\mu_F(j) = \sum_{k=1}^{n} w_k(\mathbf{a})\, c_{jk} = (C\, w(\mathbf{a}))_j. \tag{10}$$

(iii) *If* $w \colon \mathcal{A} \to \Delta_{n-1}$ *has image containing a nonempty open subset of* $\operatorname{int}(\Delta_{n-1})$*, then* $C\, w = w$ *on that set forces* $C = I$*.*

Proof. For (i) and (ii), since ν_k does not depend on $\mathbf{a}$, neither does $c_{jk} \coloneqq \nu_k[F = j]$. Applying F to the mixture (9) gives (10).

Part (iii). Let $L = C - I$. By hypothesis, $Lw = 0$ for every w in a nonempty open subset U of $\operatorname{int}(\Delta_{n-1})$. Pick $w_0 \in U$. Let $V \coloneqq \{x \in \mathbb{R}^n : \sum_j x_j = 0\}$. Here openness is relative to the affine hyperplane $\sum_j w_j = 1$, whose translation space is precisely V. Because U is open in $\operatorname{int}(\Delta_{n-1})$, there exists $r > 0$ such that $w_0 + x \in U$ for all $x \in V$ with $\|x\| < r$. For each such x, $L(w_0 + x) = 0$ and $Lw_0 = 0$, hence $Lx = 0$. By linearity, L vanishes on all of V. For each k, ℓ, $e_k - e_\ell \in V$, so $Ce_k - Ce_\ell = e_k - e_\ell$. Hence all columns satisfy $Ce_k = e_k + u$ for the same $u \in V$. For $j \neq k$, the entry $(Ce_k)_j = u_j \geq 0$ (since C is entrywise non-negative), while $\sum_j u_j = 0$. Non-negative reals summing to zero are all zero, so $u = 0$ and $C = I$. (See also Appendix B.) □

Without the open-subset hypothesis, a nonidentity C may agree with the Born map on a lower-dimensional fixed-point set (Corollary 2.6). Thus the open-subset condition is the nontriviality hypothesis in (iii).

Corollary 2.6. *Assume* $n \geq 2$*. For any selector with* $C \neq I$*, the set* $\{w \in \Delta_{n-1} : Cw = w\}$ *has zero* $(n-1)$*-dimensional Lebesgue measure relative to the affine hyperplane* $\sum_j w_j = 1$*.*

Proof. The fixed-point set $\{w \in \Delta_{n-1} : Cw = w\}$ is an affine subset of the simplex hyperplane. If it were all of Δ_{n-1}, then $Cw = w$ on $\operatorname{int}(\Delta_{n-1})$, so $C = I$ by Proposition 2.5(iii), contrary to assumption. Hence it is a proper affine subset and has zero $(n-1)$-dimensional Lebesgue measure. □

2.6 The no-go theorem

Theorem 2.7 (No-go for pointer-commuting selectors)**.** *Under* (A1)–(A5)*, let* $\mathbf{a} = (a_1, \ldots, a_{d_S})$ *range over the amplitude sphere* $\mathcal{A} = \{\mathbf{a} \in \mathbb{C}^{d_S} : \sum_k |a_k|^2 = 1\}$*. For each branch* k*, let* R_k *be the branch-conditioned record variable induced from the sector-evolved apparatus–environment state* $|\varepsilon_k(t)\rangle$ *by a fixed amplitude-independent measurable record map, taking values in a common measurable record space* $\mathbf{D}$*. Then:*

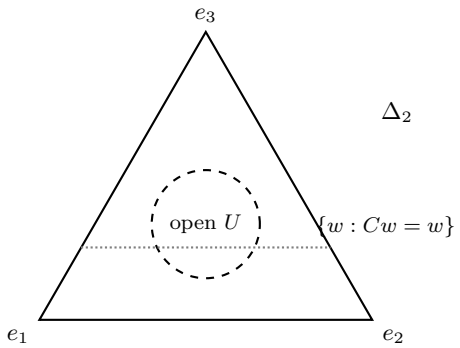

Figure 4: The probability simplex Δ_2 for $n = 3$. A non-identity column-stochastic C has fixed points on at most a proper affine subset (dotted line), which cannot contain a nonempty open subset of the simplex interior. Reproducing Born statistics on any set whose weight image contains an open interior subset forces $C = I$.

(i) *The conditional law ν_k of R_k is independent of* $\mathbf{a}$.

(ii) *Any measurable selector $F\colon \mathbf{D} \to \{1, \ldots, d_S\}$ gives an output distribution*

$$\mu_F(j) = (C\, w(\mathbf{a}))_j, \tag{11}$$

where $w_k(\mathbf{a}) = |a_k|^2$ and $C = (c_{jk})$ is a fixed column-stochastic matrix.

(iii) *Within the selector class of (A5), no such selector can reproduce the Born distribution $\mu_F(j) = |a_j|^2$ on any set whose Born-weight image contains a nonempty open subset of* $\mathrm{int}(\Delta_{d_S-1})$, *unless $C = I$.*

(iv) *$C = I$ holds if and only if there exists a measurable partition $\{D_k\}$ of $\mathbf{D}$, up to ν_k-null sets, such that $\nu_k(D_k) = 1$ for each k.*

Proof. For (i), by Lemma 2.3, $|\varepsilon_k(t)\rangle$ is independent of $\mathbf{a}$. The record map and its parameters are amplitude-independent by hypothesis, so the induced law ν_k is independent of $\mathbf{a}$. Part (ii) follows from Proposition 2.5 with $n = d_S$ and $w_k(\mathbf{a}) = |a_k|^2$. For (iii), the hypothesis provides a set $S \subseteq \mathcal{A}$ whose Born-weight image $w(S)$ contains a nonempty open subset $U \subset \mathrm{int}(\Delta_{d_S-1})$. On U, $Cw = w$ must hold if μ_F equals the Born distribution. Proposition 2.5(iii) then forces $C = I$.

For (iv), $C = I$ is equivalent to $c_{jk} = \delta_{jk}$ for all j, k, that is, for the given selector F, $\nu_k(F^{-1}(\{k\})) = 1$ for each k. Setting $D_k = F^{-1}(\{k\})$, the sets $\{D_k\}$ partition $\mathbf{D}$ and $\nu_k(D_k) = 1$. Conversely, suppose there exist measurable sets $D_1, \ldots, D_{d_S} \subseteq \mathbf{D}$ that are pairwise disjoint and whose union is $\mathbf{D}$, after modification on ν_k-null sets for each k, such that $\nu_k(D_k) = 1$ for every k. Then the selector $F = \sum_k k\, \mathbf{1}_{D_k}$ gives $c_{jk} = \nu_k(F^{-1}(\{j\})) = \delta_{jk}$, hence $C = I$. □

Corollary 2.8 (Amplitude-independence closure)**.** *Under* (A1)–(A5), *no post-processing strategy escapes the $C\, w(\mathbf{a})$ structure:*

(i) Multiple selectors. *The joint law of $(F_1, F_2, \ldots, F_m)$ applied to the same record is $P(F_1 = j_1, \ldots, F_m = j_m) = \sum_k w_k(\mathbf{a})\, T_{(j_1,\ldots,j_m),\,k}$ with fixed T.*

(ii) Adaptive strategies. *If F_2 depends on the output of F_1, the branch-conditional joint law $\nu_k(F_1 = j, F_2 = l)$ remains amplitude-independent.*

(iii) Repeated preparations. *N independent copies yield empirical frequencies converging to $C\, w(\mathbf{a})$, not $w(\mathbf{a})$. The gap does not shrink with N when $C \neq I$.*

(iv) Continuous-valued selectors. *For any measurable* $F\colon \mathbf{D} \to X$*, the law* $\mu_F(B \mid \mathbf{a}) = \sum_k w_k(\mathbf{a})\, \nu_k(F \in B)$ *is a mixture of fixed measures.*

Proof. A measurable function of amplitude-independent data is amplitude-independent. The branch-conditional law ν_k does not depend on $\mathbf{a}$ (Lemma 2.3), so any measurable functional of records drawn from ν_k inherits this independence. Parts (i)–(iv) follow by applying this observation to joint selectors, adaptive compositions, product measures, and general measurable maps, respectively. □

Remark 2.9. The branch index k labels the conditional record laws in the proof; it is *not* an input to the selector, consistent with (A5). The selector may remix the Born weights but cannot introduce new dependence on $\mathbf{a}$. Whether the partition condition $\nu_k(D_k) = 1$ holds depends on the sector Hamiltonians h_k, the initial apparatus–environment state, and the chosen record map. It is not determined by (A1)–(A5) alone.

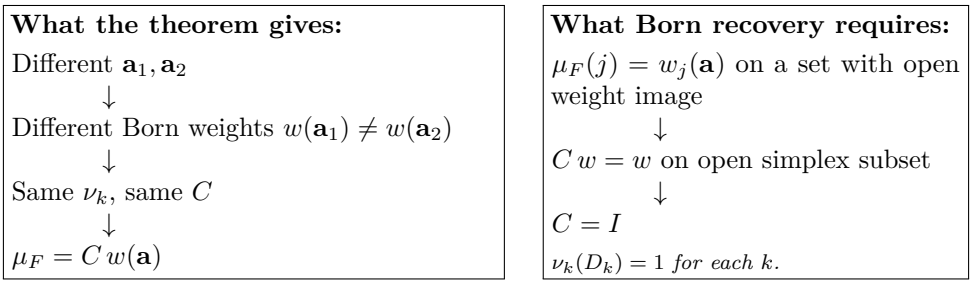

Figure 5: The no-go in two panels. Left: the selector output is always a fixed linear transformation of the Born weights. Right: reproducing Born statistics on any set whose Born-weight image contains a nonempty open subset of the simplex interior forces $C = I$, which requires $\nu_k(D_k) = 1$ for each k.

3 Supporting Closures

The results in this section sharpen, extend, or contextualize the main theorem, but none is required for Theorem 2.7; each is clearly scoped. Two tiers appear in this section. *Proved theorems* (frozen diagonal, reinstatement, conditional stability) hold under the stated assumptions. *Conditional results* whose conclusions depend on hypothesis (13) (Born-mismatch persistence) have their scope delimited by that hypothesis, which is established in the present finite-dimensional perturbative setting by Proposition 3.7.

3.1 Frozen diagonal

Corollary 3.1 (Frozen diagonal)**.** *Under* (A1)–(A4)*, the pointer-basis occupation probabilities are constant:*

$$p_k(t) \coloneqq \langle v_k|\rho_S(t)|v_k\rangle = |a_k|^2 \qquad \text{for all } t, \tag{12}$$

where $\rho_S(t) = \mathrm{Tr}_{AE}\, |\Psi(t)\rangle\langle\Psi(t)|$.

Proof. From (6), $\langle v_j|\rho_S(t)|v_k\rangle = a_j\overline{a_k}\,\langle\varepsilon_k(t)|\varepsilon_j(t)\rangle$. Setting $j = k$ gives $p_k(t) = |a_k|^2\|\varepsilon_k(t)\|^2 = |a_k|^2$, since $e^{-ih_k t}$ is unitary. □

Remark 3.2. The frozen diagonal is *not* a selector output; p_k is a diagonal element of the reduced system density matrix. Corollary 3.1 says the pointer-commuting Hamiltonian preserves the Born weights in the system state, while simultaneously producing amplitude-independent branch records. The Born weights are present in the global state but inaccessible to selectors in the class defined by (A5).

Theorem 3.3 (Basis-covariant reinstatement under primed hypotheses)**.** *Let* $\{|v'_k\rangle\}$ *be any orthonormal basis of* $\mathcal{H}_S$ *satisfying* $[H, P'_k] = 0$ *where* $P'_k = |v'_k\rangle\langle v'_k| \otimes I_{AE}$*.*

1. *Under* (A1)–(A3) *and the primed pointer-commuting condition above, the sector decomposition (Lemma 2.2) holds in the primed basis:* $H = \sum_k |v'_k\rangle\langle v'_k| \otimes h'_k$*.*
2. *If in addition a primed premeasurement* (A4′) *is given (i.e., there exists a unitary* U'_{SA} *with* $U'_{SA}\colon |v'_k\rangle|0'\rangle_A \to |v'_k\rangle|k'\rangle_A$ *producing a post-premeasurement state* $|\Psi'_0\rangle = \sum_k a'_k|v'_k\rangle|k'\rangle_A|\lambda\rangle_E$*), then Lemma 2.3 holds in the primed basis.*
3. *If further* (A5) *is applied to the primed records, then Theorem 2.7 holds in the primed basis.*

Proof. Part (i): $[H, P'_k] = 0$ gives the sector decomposition by Lemma 2.2 applied to the primed basis. Part (ii): with the primed premeasurement producing the post-premeasurement state $|\Psi'_0\rangle$, the proof of Lemma 2.3 applies verbatim with h'_k in place of h_k. Part (iii): Theorem 2.7 follows by the same argument applied to the primed records. □

3.2 Robustness under perturbation

Theorem 3.5 and its corollary are conditional results; their scope is delimited by hypothesis (13).

The exact no-go assumes perfect premeasurement and exact pointer-commutation. The following theorem is conditional: given a uniform closeness hypothesis on the perturbed branch laws, the Born mismatch persists. For a fixed selector F, define the perturbed coefficients $c^{(\epsilon)}_{jk}(\mathbf{a}) := \nu^{(\epsilon)}_k(\mathbf{a})[F = j]$ and the perturbed output law $\mu^{(\epsilon)}_F(j; \mathbf{a}) := \sum_k w_k(\mathbf{a})\, c^{(\epsilon)}_{jk}(\mathbf{a})$.

Lemma 3.4 (Positive gap)**.** *If* $C \neq I$ *is a column-stochastic matrix, then there exist an open set* $U_w \subset \mathrm{int}(\Delta_{n-1})$ *and* $m_0 > 0$ *such that* $\|C\,w - w\|_1 \geq m_0$ *for all* $w \in U_w$*. In particular, setting* $U_{\mathbf{a}} := w^{-1}(U_w) \subset \mathcal{A}$*, the bound holds for all* $\mathbf{a}$ *with* $w(\mathbf{a}) \in U_w$*.*

Proof. Choose k and j with $c_{jk} \neq \delta_{jk}$. For sufficiently small $\xi > 0$, set $w_0 = (1 - (n-1)\xi)\, e_k + \xi \sum_{m \neq k} e_m \in \mathrm{int}(\Delta_{n-1})$. Then $(C - I)w_0 \neq 0$. Continuity of $w \mapsto (C - I)w$ gives an open neighborhood U_w of w_0 in $\mathrm{int}(\Delta_{n-1})$ and $m_0 > 0$ with $\|(C - I)w\|_1 \geq m_0$ for all $w \in U_w$. □

Theorem 3.5 (Born-mismatch persistence under uniform branch-law closeness)**.** *Let* (A1)–(A3) *hold with pointer-commuting post-premeasurement dynamics. Let* $\nu^{(0)}_k$ *denote the exact branch-conditioned record laws and* $\nu^{(\epsilon)}_k(\mathbf{a})$ *the perturbed laws. Fix* $\eta > 0$ *and let* $\mathcal{A}_\eta := \{\mathbf{a} \in \mathcal{A} : |a_k|^2 \geq \eta\ \forall k\}$*. Assume the uniform closeness hypothesis:*

$$\sup_{\mathbf{a} \in \mathcal{A}_\eta} \left\|\nu^{(\epsilon)}_k(\mathbf{a}) - \nu^{(0)}_k\right\|_{\mathrm{TV}} \leq L_k\, \epsilon \quad \textit{for each } k. \tag{13}$$

Then for any measurable selector F*, the perturbed output law satisfies*

$$\sup_{\mathbf{a} \in \mathcal{A}_\eta} \left\|\mu^{(\epsilon)}_F(\mathbf{a}) - \mu^{(0)}_F(\mathbf{a})\right\|_{\mathrm{TV}} \leq L\epsilon, \tag{14}$$

where $L = \max_k L_k$*. Let* $C^{(0)}$ *be the selector matrix induced by the exact branch laws* $\nu^{(0)}_k$*, i.e.,* $c^{(0)}_{jk} = \nu^{(0)}_k[F = j]$*. If the exact output law satisfies*

$$\left\|\mu^{(0)}_F(\mathbf{a}) - w(\mathbf{a})\right\|_{\mathrm{TV}} \geq m_0 \quad \textit{for all } \mathbf{a} \in U \tag{15}$$

for some $m_0 > 0$ *and open set* $U \subseteq \mathcal{A}_\eta$ *(guaranteed by Lemma 3.4 whenever* $C^{(0)} \neq I$*), then for all* $\epsilon < m_0/L$ *the mismatch persists on* U*.*

Proof. The perturbed output law is

$$\mu_F^{(\epsilon)}(j;\mathbf{a}) = \sum_k w_k(\mathbf{a})\, c_{jk}^{(\epsilon)}(\mathbf{a}),$$

so

$$\mu_F^{(\epsilon)}(j;\mathbf{a}) - \mu_F^{(0)}(j;\mathbf{a}) = \sum_k w_k(\mathbf{a})(c_{jk}^{(\epsilon)}(\mathbf{a}) - c_{jk}^{(0)}).$$

For each k, pushforward contractivity gives $|c_{jk}^{(\epsilon)}(\mathbf{a}) - c_{jk}^{(0)}| \le \|\nu_k^{(\epsilon)}(\mathbf{a}) - \nu_k^{(0)}\|_{\mathrm{TV}} \le L_k\epsilon$. Since $w_k \ge 0$ and $\sum_k w_k = 1$, taking the $\frac{1}{2}\ell^1$ norm over j and using convexity yields $\|\mu_F^{(\epsilon)}(\mathbf{a}) - \mu_F^{(0)}(\mathbf{a})\|_{\mathrm{TV}} \le L\epsilon$ uniformly on $\mathcal{A}_\eta$. The triangle inequality gives $\|\mu_F^{(\epsilon)}(\mathbf{a}) - w(\mathbf{a})\|_{\mathrm{TV}} \ge m_0 - L\epsilon > 0$ for $\mathbf{a} \in U$ whenever $\epsilon < m_0/L$. □

Remark 3.6. Under imperfect premeasurement, the perturbed coefficients $c_{jk}^{(\epsilon)}(\mathbf{a})$ depend on $\mathbf{a}$ — only the exact $c_{jk}^{(0)}$ is amplitude-independent. The supremum over $\mathcal{A}_\eta$ absorbs this dependence.

The robustness claim is conditional on hypothesis (13), and Proposition 3.7 verifies that hypothesis within the present finite-dimensional perturbative setting. Theorem 3.5 is a stability result, not a strengthening of the no-go: it says small perturbations cannot remove the Born mismatch on U under hypothesis (13). The restriction to $\mathcal{A}_\eta$ ($|a_k|^2 \ge \eta > 0$) is required because conditioning on zero-probability sectors is undefined. Sectors with $a_k = 0$ carry zero Born weight.

3.3 General Lipschitz bound

Theorem 3.5 is conditional on hypothesis (13). The following proposition establishes that hypothesis for finite-dimensional perturbations $H_\delta = H_0 + \delta V$ with $[H_0, P_k] = 0$ under the stated assumptions.

Proposition 3.7 (General Lipschitz bound)**.** *Let* (A1)–(A3) *hold. Let* $H_\delta = H_0 + \delta V$ *with* $[H_0, P_k] = 0$ *and* V *Hermitian. Fix* $\eta > 0$ *and let* $\mathcal{A}_\eta := \{\mathbf{a} \in \mathcal{A} : |a_k|^2 \ge \eta\ \forall k\}$*. Then for each* k*, the perturbed branch-conditioned record laws satisfy*

$$\sup_{\mathbf{a}\in\mathcal{A}_\eta} \|\nu_k^{(\delta)}(\mathbf{a}) - \nu_k^{(0)}\|_{\mathrm{TV}} \le \delta\, t \left[\|P_k V P_k\| + \|[V, P_k]\|\sqrt{\frac{1-\eta}{\eta}}\right]. \tag{16}$$

The estimate applies to Hermitian perturbations V *under the stated hypotheses.*

Proof. The proof proceeds in four steps.

Step A (Dyson decomposition). Work in the interaction picture. To first order in δ,

$$P_k|\psi^{(\delta)}(t)\rangle = a_k|\varepsilon_k^{(0)}(t)\rangle - i\delta \int_0^t e^{-iH_0(t-s)}\, P_k V\, e^{-iH_0 s}\, |\psi_0\rangle\, ds + O(\delta^2).$$

Since $[H_0, P_k] = 0$, insert $I = P_k + (I - P_k)$ after V: $P_k V = P_k V P_k + P_k V(I - P_k)$. The $P_k V P_k$ term (blind) acts within sector k, producing a correction proportional to a_k. The $P_k V(I - P_k)$ term (leakage) pulls amplitude from sectors $j \ne k$, producing corrections proportional to a_j.

Step B (Commutator identity). The commutator $[V, P_k]$ has anti-block-diagonal structure:

$$[V, P_k] = \begin{pmatrix} 0 & -B^\dagger \\ B & 0 \end{pmatrix}, \qquad B = (I - P_k)V P_k,$$

so $\|[V, P_k]\| = \|B\| = \|B^\dagger\| = \|P_k V(I - P_k)\|$.

Step C (Leakage estimate). The total leakage at time s acts on $|\chi(s)\rangle = (I - P_k)\, e^{-iH_0 s}\, |\psi_0\rangle$, which has norm $\sqrt{1 - |a_k|^2}$ (since $[H_0, P_k] = 0$ and $\|P_k|\psi_0\rangle\| = |a_k|$). Therefore

$$\|P_k V (I - P_k)\, |\chi(s)\rangle\| \leq \|[V, P_k]\| \sqrt{1 - |a_k|^2} \leq \|[V, P_k]\| \sqrt{1 - \eta}.$$

Integrating over $[0, t]$ and dividing by $\|P_k|\psi^{(\delta)}\rangle\| \geq |a_k| \geq \sqrt{\eta}$ gives leakage contribution $\leq \delta\, t\, \|[V, P_k]\| \sqrt{(1-\eta)/\eta}$. The blind contribution is bounded by $\delta\, t\, \|P_k V P_k\|$ by the same integration.

Step D (State distance to TV distance). For pure states, $D(\rho, \sigma) \leq \|\psi - \phi\|$. The data-processing inequality gives $\|\nu_k^{(\delta)} - \nu_k^{(0)}\|_{\mathrm{TV}} \leq D(\rho_k^{(\delta)}, \rho_k^{(0)})$. Combining Steps A–C with Step D yields the bound. □

Corollary 3.8 (General mismatch persistence)**.** *Under Proposition 3.7, combined with Lemma 3.4 (gap $m_0 > 0$ whenever $C^{(0)} \neq I$) and Theorem 3.5: the Born mismatch persists for all*

$$\delta < \frac{m_0}{t \left[\|P_k V P_k\| + \|[V, P_k]\| \sqrt{\frac{1-\eta}{\eta}} \right]}.$$

The estimate applies to Hermitian perturbations V under the stated hypotheses.

Proof. Substitute the bound from Proposition 3.7 into Theorem 3.5 with $L = \max_k L_k$. □

Corollary 3.9 (Pointer-commuting stability)**.** *When $[V, P_k] = 0$, the leakage term vanishes identically. The perturbed record laws $\nu_k^{(\delta)}$ remain amplitude-independent, and the no-go (Theorem 2.7) extends to all perturbations within the pointer-commuting class.*

Proof. $[V, P_k] = 0$ forces the leakage term in Proposition 3.7 to zero. The blind correction is proportional to a_k and cancels in the conditional state normalization, so the perturbed record laws $\nu_k^{(\delta)}$ remain amplitude-independent. Theorem 2.7 then applies to the perturbed dynamics. □

Remark 3.10 (Verified models). Proposition 3.7 is verified symbolically (SymPy) for $d_S = d_A = d_E = 2$ models (2300 random trials, 0 violations) and $d_S = d_A = d_E = 3$ models (200 random trials, 0 violations), as well as non-rank-1 projectors (3000 random trials, 0 violations). The general bound subsumes the constants previously obtained in four explicit $2 \times 2 \times 2$ models. In the tested regimes, the worst-case ratio actual/bound approaches 1 as $\delta \to 0$; this is empirical support and does not strengthen the analytic theorem. Details of the symbolic verification are described in Appendix C.

4 Deterministic and Stochastic Trajectories

Theorem 2.7 shows that deterministic measurable selectors restricted to pointer-commuting branch records cannot reproduce Born statistics under (A1)–(A5). This section makes that obstruction concrete by comparing two dynamical regimes. Under deterministic pointer-commuting evolution, $p_k(t)$ is constant (Corollary 3.1); in the binary SSE benchmark, $p_0(t)$ is a martingale with state-dependent diffusion that converges almost surely to a vertex in the binary case considered below. For general d_S, simplex preservation and the martingale property extend (Remark 4.3). The comparison is structural: in the SSE benchmark, the outcome-probability process has nonzero state-dependent volatility, which the deterministic regime lacks.

4.1 Frozen branch weights

The frozen-diagonal theorem (Corollary 3.1) establishes that under (A1)–(A4) the pointer-basis occupation probabilities satisfy $p_k(t) = |a_k|^2$ for all t. Viewed as a process, this means the deterministic pointer-commuting regime produces the trivial dynamics $dp_k = 0$: the Born weights are present in the global state but undergo no dynamics. Within the present selector class, no stopping time or deterministic rule applied to branch records alters this, because the branch records are amplitude-independent (Lemma 2.3).

4.2 Stochastic trajectory model

We introduce a canonical stochastic model for comparison. The setup is minimal: a continuous-monitoring SSE with a Lindblad operator diagonal in the pointer basis. The purpose is to exhibit a concrete model in which $p_k(t)$ is *not* frozen, not to advocate any particular interpretation.

Setup. Let $d_S = 2$ with $|\psi(t)\rangle = \alpha(t)|v_0\rangle|0\rangle_A + \beta(t)|v_1\rangle|1\rangle_A$, $|\alpha|^2 + |\beta|^2 = 1$, and $p_0 = |\alpha|^2$. Consider the nonlinear SSE with Lindblad operator $L = I_S \otimes \sigma_z^A$ (diagonal in the pointer basis, eigenvalues ± 1) and monitoring rate $\gamma > 0$:

$$d|\psi\rangle = -\tfrac{\gamma}{2}(L - \langle L\rangle)^2|\psi\rangle\, dt + \sqrt{\gamma}\,(L - \langle L\rangle)|\psi\rangle\, dW, \tag{17}$$

where $\langle L\rangle = \langle\psi|L|\psi\rangle = 2p_0 - 1$ and W is a standard Wiener process.

Proposition 4.1 (Martingale dynamics)**.** *Under* (17)*, $p_0(t)$ satisfies*

$$dp_0 = 4\sqrt{\gamma}\, p_0(1 - p_0)\, dW. \tag{18}$$

In particular:

1. *p_0 is a driftless diffusion (martingale) with state-dependent volatility $\sigma(p_0) = 4\sqrt{\gamma}\, p_0(1-p_0)$.*
2. *$\mathbb{E}[p_0(t)] = |a_0|^2$ for all $t \geq 0$.*
3. *$p_0(t) \to p_\infty \in \{0, 1\}$ almost surely, with $\Pr(p_\infty = 1) = |a_0|^2$.*

Proof. Since L is diagonal with entries ± 1 in the premeasurement subspace, one computes $(L - \langle L\rangle)|\psi\rangle = 2(1 - p_0)\alpha|v_0 0_A\rangle - 2p_0\beta|v_1 1_A\rangle$. Applying Itô's formula to $p_0 = |\alpha|^2$: the drift terms cancel exactly ($-4\gamma p_0(1 - p_0)^2 + 4\gamma p_0(1 - p_0)^2 = 0$), leaving the diffusion (18). Since $p_0(t) \in [0, 1]$, the process is integrable; the martingale property then gives $\mathbb{E}[p_0(t)] = p_0(0) = |a_0|^2$ for all $t \geq 0$, establishing item (ii).

For item (iii), we establish convergence to vertices via a quadratic variation argument. Since $0 \leq p_0(t) \leq 1$, the martingale is uniformly bounded in L^2. For the continuous martingale $M_t := p_0(t) - p_0(0)$,

$$\mathbb{E}[M_t^2] = \mathbb{E}[\langle M\rangle_t] = \mathbb{E}[\langle p_0\rangle_t].$$

Hence $\mathbb{E}[\langle p_0\rangle_t] = \mathbb{E}[p_0(t)^2] - p_0(0)^2 \leq 1$ for all t. By monotone convergence,

$$\langle p_0\rangle_\infty = \int_0^\infty 16\gamma\, p_0(s)^2(1 - p_0(s))^2\, ds < \infty \quad \text{a.s.} \tag{19}$$

By the bounded martingale convergence theorem, $p_0(t) \to p_\infty \in [0, 1]$ a.s. Since $p_0(t)$ has continuous sample paths and $p_0(t) \to p_\infty$ a.s., on the event $\{p_\infty \in (0, 1)\}$ there exists $T < \infty$ such that for all $s \geq T$, $p_0(s) \geq p_\infty/2$ and $1 - p_0(s) \geq (1 - p_\infty)/2$. Then $p_0(s)^2(1 - p_0(s))^2 \geq p_\infty^2(1 - p_\infty)^2/16 > 0$ for all $s \geq T$, making the integral in (19) diverge, a contradiction. Hence $p_\infty \in \{0, 1\}$ a.s. Since p_∞ is $\{0, 1\}$-valued and $\mathbb{E}[p_\infty] = |a_0|^2$, we have $\Pr(p_\infty = 1) = |a_0|^2$. □

Remark 4.2 (Normalization convention). The factor 4 in (18) arises because $L = \sigma_z^A$ has eigenvalue gap $l_0 - l_1 = 2$; conventions with $L = \sigma_z^A/2$ yield $2\sqrt{\gamma}$ instead.

Remark 4.3 (General d_S). For arbitrary $d_S \geq 2$ with Lindblad operator $L = I_S \otimes L_A$, $L_A|k\rangle_A = l_k|k\rangle_A$ (distinct eigenvalues), the same calculation yields

$$dp_k = 2\sqrt{\gamma}\, p_k(l_k - \langle L\rangle)\, dW, \qquad \langle L\rangle = \sum_j p_j l_j. \tag{20}$$

Here dW is a single standard Wiener process (one measurement channel); simplex preservation $\sum_k dp_k = 0$ follows immediately, and the martingale property extends directly.

4.3 Comparison

The two regimes can now be compared directly.

Proposition 4.4 (Deterministic–stochastic comparison)**.**

1. Deterministic (pointer-commuting H). *Under (A1)–(A4), the branch-weight process satisfies $dp_k = 0$ for all k. The outcome probabilities are frozen at their initial values $p_k(0) = |a_k|^2$.*

2. Stochastic (SSE with diagonal L, binary case). *Under (17) with $d_S = 2$, $dp_0 = \sigma(p_0)\, dW$ with state-dependent $\sigma(p_0) \neq 0$ at interior points. The process converges almost surely to $p_\infty \in \{0, 1\}$ with $\Pr(p_\infty = 1) = |a_0|^2$ (Proposition 4.1).*

3. Structural gap. *The deterministic pointer-commuting branch-weight process is the trivial process $dp_k = 0$: no runwise variability occurs. In the SSE benchmark, the outcome-probability process has nonzero state-dependent volatility $\sigma(p) \neq 0$ at every interior point of the simplex. In this SSE model, the volatility arises from the backaction term $\sqrt{\gamma}(L - \langle L\rangle)|\psi\rangle\, dW$, which couples to the current branch weights through $\langle L\rangle$.*

Proof. Part (i) is Corollary 3.1. Part (ii) is Proposition 4.1 (convergence is proved for $d_S = 2$; Remark 4.3 extends the SDE form to general d_S). Part (iii) follows by comparison: Theorem 2.7 shows that deterministic selectors on amplitude-independent records produce output $\mu_F = C\, w(\mathbf{a})$ for fixed C, whereas the stochastic process has nonzero state-dependent volatility at every interior point of the simplex. □

4.4 Escape channels

The no-go proved here does not exclude mechanisms that supply amplitude information through channels outside the branch-conditioned record space considered in (A5). Representative routes to Born recovery introduce amplitude sensitivity through structures outside (A1)–(A5):

Escape channel	**Assumption relaxed**	**Mechanism**
Stochastic trajectory	Leaves the deterministic pointer-commuting regime	State-dependent stochastic backaction
Hidden-variable measure	Introduces amplitude-sensitive auxiliary variables outside (A5)	$\|\psi\|^2$-equivariant configuration
Objective collapse	Non-unitary dynamics	Stochastic reduction law
Branch-mixing dynamics	(A3)	Cross-sector coupling

Table 1: Escape channels. This taxonomy is descriptive rather than exhaustive.

4.5 Scope of the comparison

Remark 4.5 (Interpretive boundaries). This section does not claim that stochastic dynamics is the only route to Born-rule outcome selection. Bohmian mechanics achieves selection through $|\psi|^2$-equivariant hidden-variable distributions [3, 5]; objective-collapse theories [1, 6] modify the dynamical law. Both lie outside (A1)–(A5) and are not addressed here. In the canonical SSE benchmark used here, the outcome-probability process has state-dependent diffusion, which is absent from the deterministic pointer-commuting regime. It is a comparison of mathematical structures, not a claim about physical reality. The SSE section is included only as a structural contrast and is not part of the proof of Theorem 2.7.

5 Discussion

5.1 What the no-go result establishes

Theorem 2.7 establishes a scoped obstruction: under (A1)–(A5), deterministic measurable selectors restricted to branch-conditioned apparatus–environment records cannot reproduce Born statistics on any set whose Born-weight image contains a nonempty open subset unless the associated column-stochastic matrix is the identity. The contribution is an explicit decomposition of this obstruction: within this regime, the only route to Born-matching outcome selection is perfect branch identification from the record (Theorem 2.7(iv)); the achievability of this condition is analyzed in Section 5.2.

5.2 When is Born recovery achievable?

Theorem 2.7(iv) provides a complete characterization: Born recovery holds if and only if the conditional record laws ν_k admit a measurable partition $\{D_k\}$ with $\nu_k(D_k) = 1$ for each k. This is not merely an impossibility result; within (A1)–(A5), it identifies the exact structural requirement.

For the optimal record map, the partition condition reduces to mutual orthogonality of the sector-evolved states $\{|\varepsilon_k(t)\rangle\}$: orthogonal pure states are perfectly distinguishable by some POVM, while non-orthogonal pure states are not [7]. By premeasurement, the initial apparatus pointers are orthonormal: $(\langle j|_A \otimes \langle\lambda|_E)(|k\rangle_A \otimes |\lambda\rangle_E) = \delta_{jk}$, so the sector-evolved states start orthogonal. Whether they remain so depends on the sector Hamiltonians. When $h_j = h_k$ for all j, k (system-decoupled dynamics), the same unitary acts on every sector and orthogonality is preserved for all t. In the two explicit models of Remark 3.10, the inter-branch overlap vanishes identically; the partition condition is satisfied, and Born recovery is achievable by measuring the apparatus–environment state in any basis that distinguishes the sector-evolved states.

For general pointer-commuting Hamiltonians with $h_j \neq h_k$, the relative evolution operator $G(t) = e^{+ih_j t}e^{-ih_k t}$ can rotate $|k\rangle_A$ toward $|j\rangle_A$, producing nonzero overlap $\langle\varepsilon_j(t)|\varepsilon_k(t)\rangle \neq 0$. When this occurs, no record map achieves the partition condition, and Born recovery is impossible for every deterministic selector within the class of (A5).

Structural stability of the partition condition

The overlap $f(t) = \langle\psi|G(t)|\phi\rangle$ is an exponential polynomial $f(t) = \sum_{m,l} C_{ml}\, e^{i(E_m^{(j)} - E_l^{(k)})t}$ with at most $(d_A d_E)^2$ terms. Three levels of conditions guarantee $f(t) = 0$, each strictly weaker than the previous.

Theorem 5.1 (Structural stability of the partition condition)**.** *Under* (A1)–(A3)*, the following are equivalent for generic sector Hamiltonians (those with no accidental eigenvalue resonances between distinct sectors):*

(i) *The partition condition* $\nu_k(D_k) = 1$ *holds for all* $|\lambda\rangle_E$.

(ii) *The relative evolution* $G(t) = e^{ih_j t}\, e^{-ih_k t}$ *is block-diagonal in the pointer decomposition for all* t.

(iii) *The sector Hamiltonians* h_j, h_k *are pointer-commuting (block-diagonal in the apparatus basis).*

For non-generic sector Hamiltonians, (i) can hold for specific $|\lambda\rangle_E$ *via Krylov orthogonality or spectral cancellation, but these conditions are structurally unstable (destroyed by infinitesimal perturbation of the Hamiltonians).*

Proof. (iii)⇒(ii)⇒(i) is immediate. For (i)⇒(iii) under genericity: the overlap $f(t)$ is an almost-periodic function. By analyticity, $f(t) = 0$ for all t iff all derivatives vanish at $t = 0$, which by the Sylvester operator $L(X) = h_j X - X h_k$ and Cayley–Hamilton requires at most $(d_A d_E)^2$ checks. For generic eigenvalues (no resonances), all spectral coefficients C_{ml} must vanish individually. The "for all $|\lambda\rangle_E$" hypothesis forces this vanishing to be structural (not dependent on initial-state overlaps), which requires block-diagonality of h_j, h_k. □

Remark 5.2. Pointer-commuting is not merely a technical convenience. Within the present framework, it is the structural condition that ensures branch-conditioned records remain distinguishable independently of the initial environment state. The stability hierarchy shows that all non-pointer-commuting routes to the partition condition require fine-tuning that has no physical protection mechanism.

Remark 5.3. For $d_A = 2$, $d_E = 1$, the overlap can be computed in closed form: $f(t) = 0$ for all t iff either (a) both h_k are pointer-diagonal, or (b) $h_j - h_k = cI$ for some $c \in \mathbb{R}$. Case (b) is non-generic and destroyed by perturbation.

Exponential-polynomial structure of exact partition

The inter-branch overlap $f_{k\ell}(t) = \langle \varepsilon_k(t) | \varepsilon_\ell(t) \rangle = \sum_j c_j\, e^{i\omega_j t}$ is an exponential polynomial whose zero set $Z(f) = \{t : f(t) = 0\}$ determines the times at which exact partition holds.

Remark 5.4 (Skolem connection). Deciding whether $Z(f)$ is nonempty for a given exponential polynomial is an instance of the continuous Skolem problem, whose decidability is open for three or more independent frequencies [4]. The quantum overlap with $d_A d_E = d$ has at most d^2 terms and $2d - 1$ independent frequencies; for $d \geq 3$ this exceeds the decidability frontier. The approximate version ($|f(t)| < \varepsilon$ for some t) is always decidable.

Corollary 5.5 (Generic finiteness)**.** *For generic sector Hamiltonians with rationally independent eigenvalue differences,* $f \not\equiv 0$ *implies* $Z(f)$ *is finite. Exact partition is temporally nongeneric: it occurs at most finitely many times, and after the last zero, the sector-evolved states are never again perfectly orthogonal.*

Remark 5.6 (Galois genericity). For sector Hamiltonians with algebraically generic eigenvalues (Galois group $S_d \times S_d$ over $\mathbb{Q}$), all d^2 frequencies are distinct, which eliminates the spectral cancellation level from the stability hierarchy: the partition condition reduces to Krylov orthogonality.

Within the selector and dynamics class of (A1)–(A5), deterministic Born recovery is possible if and only if the branch-conditioned record laws identify the branch almost surely. Corollary 3.1 shows that the Born weights remain present in the evolving state, but Lemma 2.3 and (A5) show that this amplitude information does not enter the admissible branch-record channel available to the selector.

5.3 Scope of the theorem

The theorem applies only within a precisely delimited regime. The following are in scope:

- Deterministic, pointer-commuting post-premeasurement evolution satisfying $[H, P_k] = 0$ for all k.
- Finite-dimensional tripartite Hilbert spaces $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_E$.
- Measurable selectors whose input is restricted to apparatus–environment record data, with no access to branch labels or amplitude vectors.
- Arbitrary initial environment states $|\lambda\rangle_E$.

The following are outside scope:

- Hamiltonians that do not commute with the pointer projectors. Branch-mixing dynamics ($[H, P_k] \neq 0$) allows amplitude information to leak into branch records. Proposition 3.7 provides a general perturbative bound, but the non-perturbative regime is not addressed.
- Hidden-variable theories in which outcome selection depends on variables distributed according to an amplitude-sensitive measure (for example, Bohmian mechanics with $|\psi|^2$-equivariant configuration distributions [3, 5]) are outside scope.
- Stochastic or nonlinear modifications of the dynamical law, including objective-collapse models [1, 6] and continuous-measurement unravelings [12]. These fall outside the deterministic pointer-commuting regime studied here.
- Claims about which interpretation of quantum mechanics is correct. The theorem is a mathematical result about a specific dynamical and selector class, not an interpretational claim.

Remark 5.7 (POVM structure beyond perturbation theory). For arbitrary Hamiltonians (not necessarily close to pointer-commuting), the branch weights have the exact form $p_k(\mathbf{a}, t) = \mathbf{a}^\dagger M^{(k)}(t)\,\mathbf{a}$, where $M^{(k)}_{mn}(t) = \langle\phi_m|U(t)^\dagger P_k\,U(t)|\phi_n\rangle$ and $\{M^{(k)}(t)\}_{k=1}^{d_S}$ is a POVM on amplitude space. When $[H, P_k] = 0$, each $M^{(k)}$ is diagonal, recovering the column-stochastic structure $\mu_F = C\,w(\mathbf{a})$. When $[H, P_k] \neq 0$, off-diagonal entries encode interference terms. A deterministic selector F achieves Born recovery iff $\sum_{k:\,F(k)=j} M^{(k)} = |j\rangle\langle j|$ for all j. Since each $M^{(k)}$ is positive semidefinite, this forces $M^{(k)} = \lambda_k |F(k)\rangle\langle F(k)|$, a generalized sector-confinement condition that reduces to pointer-commuting on the relevant subspace. Non-commuting dynamics do not circumvent this obstruction.

5.4 Decomposition as contribution

The paper's primary contribution is an explicit decomposition of the measurement-selection problem into typed components with clear dependency structure:

1. *Sector preservation* (Lemma 2.2): pointer-commuting dynamics confines evolution to independent sectors.
2. *Amplitude independence* (Lemma 2.3): sector-evolved states and their record laws do not depend on system amplitudes.
3. *Linear remixing* (Proposition 2.5): any deterministic selector on amplitude-independent records produces output that is a fixed column-stochastic transformation of the Born weights.

4. *Rigidity* (Proposition 2.5(iii)): matching Born statistics on any set with open Born-weight image forces this transformation to be the identity.

The value of the decomposition is that it makes the dependency structure explicit and shows exactly where any deterministic account within this framework must depart from the chain above.

5.5 Relation to decoherence and trajectory models

Decoherence and trajectory models address different parts of the measurement problem, and the present theorem sits between them. The sector decomposition (Lemma 2.2) and the amplitude-independence of branch records (Lemma 2.3) arise from the same pointer-commuting structure that appears in a broad subclass of decoherence models with pointer-diagonal couplings.

Within the present model class, decoherence-like record stabilization does not by itself supply amplitude dependence to the selector. Record formation and outcome selection remain logically distinct under (A1)–(A5).

The SSE comparison in Section 4 is included only as a structural contrast. It exhibits one mechanism (state-dependent runwise stochastic variability) not present in the deterministic pointer-commuting regime studied here. The paper does not claim that such stochasticity is necessary for Born recovery in general.

5.6 Relation to existing no-go theorems

The present result is distinct from, and complementary to, established no-go theorems in quantum foundations.

Bell-type theorems [2] constrain local hidden-variable models by showing that local realism is incompatible with quantum correlations, while the Kochen–Specker theorem [9] constrains noncontextual value assignments. Those results concern ensemble statistics and the structure of hidden-variable models. The present theorem instead constrains deterministic functionals of pointer-commuting branch records. A hidden-variable theory that supplies amplitude information through an additional channel (such as the $|\psi|^2$-equivariant configuration measure of Bohmian mechanics [3, 5]) is outside the scope of the present theorem.

5.7 Limits and future directions

The preceding results identify the remaining questions that would have to be answered to extend or physically instantiate the present framework.

Theorem 5.1 classifies the partition condition of Theorem 2.7(iv): within this framework, it holds precisely under pointer-commuting. The remaining physical question is to identify concrete physical regimes in which the sector Hamiltonians are pointer-commuting or sufficiently close to it.

The general Lipschitz bound (Proposition 3.7) shows that, within the present finite-dimensional setting, the Born mismatch persistence argument extends to perturbations $H_\delta = H_0 + \delta V$ with $[H_0, P_k] = 0$. The bound is dimension-independent and applies to the perturbative class covered by Proposition 3.7. Symbolic and numerical tests reported in Appendix C are consistent with near-tight behavior in the tested regimes, but are supplementary to the analytic proof. Corollary 3.9 shows that perturbations within the pointer-commuting class preserve amplitude-independence of the record laws exactly.

The analysis is carried out in finite dimensions for clean sector decomposition, conditioning, and measurability. The finite-dimensional proof isolates amplitude-independence of branch-conditioned records as the operative mechanism of the obstruction, but extending the argument to separable infinite-dimensional apparatus–environment models would require additional control of domains, conditioning, and measurable record maps, and is not attempted here.

The SSE comparison exhibits one mechanism (state-dependent runwise stochastic variability) not present in the deterministic pointer-commuting regime studied here. The paper does not claim that such stochasticity is necessary for Born recovery in general.

6 Conclusion

Under (A1)–(A5), any deterministic measurable selector restricted to branch-conditioned apparatus–environment records yields an output distribution $\mu_F = C\, w(\mathbf{a})$ for a fixed column-stochastic matrix C (Theorem 2.7). Agreement with the Born rule on any set whose Born-weight image contains a nonempty open subset of $\mathrm{int}(\Delta_{d_S-1})$ forces $C = I$; equivalently, the conditional record laws ν_k must admit a measurable partition $\{D_k\}$ with $\nu_k(D_k) = 1$ for each k.

The exact no-go holds throughout the class defined by (A1)–(A5). Within the present finite-dimensional setting, Proposition 3.7 and Corollary 3.8 extend the mismatch-persistence argument to perturbations $H_\delta = H_0 + \delta V$ with $[H_0, P_k] = 0$, with an explicit dimension-independent bound under the stated assumptions.

The theorem therefore identifies a precise obstruction within the selector and dynamics class of (A1)–(A5): deterministic Born matching on an open weight-space set requires almost-sure branch identification by the branch-conditioned record laws. Corollary 3.1 shows that the Born weights remain present in the evolving state, but Lemma 2.3 and (A5) show that this amplitude information does not enter the admissible branch-record channel available to the selector.

The partition condition of Theorem 2.7(iv) is classified within the present framework: Theorem 5.1 identifies pointer-commuting sector Hamiltonians as the structurally stable condition for Born recovery within this framework. The remaining physical question is to identify concrete regimes in which this structural condition is realized. The paper does not exclude theories in which amplitude information reaches the selector through channels outside this framework.

References

[1] Angelo Bassi, Kinjalk Lochan, Seema Satin, Tejinder P. Singh, and Hendrik Ulbricht. "Models of wave-function collapse, underlying theories, and experimental tests". In: *Reviews of Modern Physics* 85 (2013), pp. 471–527.

[2] John S. Bell. "On the Einstein–Podolsky–Rosen paradox". In: *Physics Physique Fizika* 1 (1964), pp. 195–200.

[3] David Bohm. "A suggested interpretation of the quantum theory in terms of "hidden" variables". In: *Physical Review* 85 (1952), pp. 166–193.

[4] Ventsislav Chonev, Joël Ouaknine, and James Worrell. "On the Skolem Problem for Continuous Linear Dynamical Systems". In: *43rd International Colloquium on Automata, Languages, and Programming (ICALP 2016)*. Vol. 55. Leibniz International Proceedings in Informatics (LIPIcs). 2016, 100:1–100:13.

[5] Detlef Dürr and Stefan Teufel. *Bohmian Mechanics: The Physics and Mathematics of Quantum Theory*. Springer, 2009.

[6] G. C. Ghirardi, A. Rimini, and T. Weber. "Unified dynamics for microscopic and macroscopic systems". In: *Physical Review D* 34 (1986), pp. 470–491.

[7] Carl W. Helstrom. *Quantum Detection and Estimation Theory*. Academic Press, 1976.

[8] Erich Joos and H. Dieter Zeh. "The emergence of classical properties through interaction with the environment". In: *Zeitschrift für Physik B* 59 (1985), pp. 223–243.

[9] Simon Kochen and Ernst P. Specker. "The problem of hidden variables in quantum mechanics". In: *Journal of Mathematics and Mechanics* 17 (1967), pp. 59–87.

[10] John von Neumann. *Mathematische Grundlagen der Quantenmechanik*. English translation: *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, 1955. Springer, 1932.

[11] Maximilian Schlosshauer. *Decoherence and the Quantum-to-Classical Transition*. Springer, 2007.

[12] Howard M. Wiseman and Gerard J. Milburn. *Quantum Measurement and Control*. Cambridge University Press, 2010.

[13] H. Dieter Zeh. "On the interpretation of measurement in quantum theory". In: *Foundations of Physics*. Vol. 1. 1970, pp. 69–76.

[14] Wojciech H. Zurek. "Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse?" In: *Physical Review D* 24 (1981), pp. 1516–1525.

[15] Wojciech H. Zurek. "Environment-induced superselection rules". In: *Physical Review D* 26 (1982), pp. 1862–1880.

[16] Wojciech H. Zurek. "Decoherence, einselection, and the quantum origins of the classical". In: *Reviews of Modern Physics* 75 (2003), pp. 715–775.

A Assumption Dependencies

For ease of reference, this appendix records the assumptions used in the paper and the results that depend on them. Assumptions (A1)–(A5) define the physical setup and selector class for the main theorem; all remaining assumptions are used only in supporting results.

Label	**Content**	**Used in**
(A1)	Finite Hilbert space	All Hilbert-space results except Proposition 2.5
(A2)	Pointer basis	All Hilbert-space results except Proposition 2.5
(A3)	Pointer-commuting dynamics	All Hilbert-space results except Proposition 2.5
(A4)	Perfect premeasurement	Lemma 2.3, Corollary 3.1, and Theorem 2.7
(A5)	Deterministic selector on branch-conditioned records	Theorem 2.7
(A8)	Primed basis with $[H, P'_k] = 0$	Theorem 3.3(i) only
(A4′)	Primed premeasurement	Theorem 3.3(ii)
(A5′)	Primed deterministic selector	Theorem 3.3(iii)
(A9)–(A11)	Binary SSE setup	Proposition 4.1 only
(A9′)–(A11′)	General-d SSE	Remark 4.3 only

The main theorem chain (Lemma 2.2 → Lemma 2.3 → Proposition 2.5 → Theorem 2.7) uses only (A1)–(A5). Proposition 2.5 is model-independent and uses no Hilbert-space assumptions; it requires only amplitude-independent conditional record laws ν_k and an unconditional law of the form $\sum_k w_k(\mathbf{a})\,\nu_k$. The robustness results (Lemma 3.4 and Theorem 3.5) use (A1)–(A3) together with the uniform branch-law closeness hypothesis (13), which replaces exact (A4). In

that regime, the perturbed branch laws $\nu_k^{(\epsilon)}(\mathbf{a})$ are assumed uniformly close in total variation to the exact amplitude-independent laws $\nu_k^{(0)}$. Proposition 3.7 establishes this hypothesis in the present finite-dimensional perturbative setting, yielding Corollary 3.8.

B Auxiliary Proof Details

This appendix records short proof details used in the main text.

Open-set rigidity for Proposition 2.5(iii)

Assume $Cw = w$ on a nonempty open subset $U \subset \mathrm{int}(\Delta_{n-1})$. Then $C = I$.

1. *L vanishes on the translation space.* Let $L = C - I$. Pick $w_0 \in U$. For x in the codimension-1 subspace $V = \{x \in \mathbb{R}^n : \sum_j x_j = 0\}$ with $\|x\|$ sufficiently small, $w_0 + x \in U$, so $Lx = L(w_0 + x) - Lw_0 = 0$. By linearity, $L|_V = 0$.

2. *The columns of C differ by a common vector.* Since $e_k - e_\ell \in V$ for all k, ℓ, we have $Ce_k - Ce_\ell = e_k - e_\ell$. Fix k_0 and set $u := Ce_{k_0} - e_{k_0}$. Then $Ce_k = e_k + u$ for every k.

3. *Entrywise non-negativity forces that vector to vanish.* For $j \neq k$, the entry $(Ce_k)_j = u_j \geq 0$ (since C is entrywise non-negative). Column-stochasticity gives $\sum_j u_j = 0$. Non-negative reals summing to zero are all zero, so $u = 0$ and $C = I$.

This argument uses only that C is column-stochastic (entrywise non-negative with unit column sums) and that the fixed-point set contains an open subset of the simplex interior. It does not use any Hilbert-space structure.

Lower-dimensional fixed-point sets in Corollary 2.6

This example shows that the open-set hypothesis in Proposition 2.5(iii) cannot be dropped. The case $C = P$ with $P \neq I$ is a special case of Corollary 2.6: the invariant set $\{w : Pw = w\}$ is a proper affine subspace of the simplex and contains no open interior subset.

Normalization in the record-law observation

The branch-conditioned record variable R_k is induced from the sector-evolved state $|\varepsilon_k(t)\rangle = e^{-ih_k t}(|k\rangle_A|\lambda\rangle_E)$ (Equation (7)) by an amplitude-independent record map. The state $|\varepsilon_k(t)\rangle$ is already normalized (since $e^{-ih_k t}$ is unitary), so no amplitude-dependent normalization factor enters. When $a_k \neq 0$, the sector-evolved state coincides with the normalized conditional state $P_k|\Psi(t)\rangle / \|P_k|\Psi(t)\rangle\|$ up to the system factor $|v_k\rangle$; the normalization $\|P_k|\Psi(t)\rangle\| = |a_k|$ cancels with the prefactor a_k in $P_k|\Psi(t)\rangle = a_k|v_k\rangle \otimes |\varepsilon_k(t)\rangle$.

Born map surjectivity

Lemma B.1 (Born-map surjectivity)**.** *The Born map $\mathbf{a} \mapsto (|a_1|^2, \ldots, |a_{d_S}|^2)$ from the amplitude sphere $\mathcal{A} = \{\mathbf{a} \in \mathbb{C}^{d_S} : \sum_k |a_k|^2 = 1\}$ to Δ_{d_S-1} is surjective.*

Proof. For any $w \in \Delta_{d_S-1}$, the vector $\mathbf{a} = (\sqrt{w_1}, \ldots, \sqrt{w_{d_S}})$ lies in $\mathcal{A}$ and maps to w. □

This surjectivity is used in Theorem 2.7(iii) to transfer the open-set condition from the simplex to the amplitude sphere.

C Symbolic Verification

The algebraic identities underlying the main results have been independently confirmed by exact symbolic computation (SymPy). These computations are supplementary consistency tests; they do not replace the mathematical arguments in the paper text. Every theorem stands on its own proof.

Scope of verification

The symbolic computation operates on explicit finite-dimensional models. The principal model is $d_S = 2$, $d_A = 2$, $d_E = 2$ (8-dimensional total Hilbert space) with two Hamiltonians:

- *System-decoupled*: $H = I_S \otimes \sigma_z^A \otimes \sigma_x^E$.
- *System-coupled*: $H = \sigma_z^S \otimes \sigma_z^A \otimes \sigma_x^E$.

Both satisfy $[H, P_k] = 0$ and have involution property $H^2 = I$, giving closed-form exponentials.

Symbolic checks performed

The following classes of identity were verified exactly in the scoped models (83 checks total; details in the supplementary SymPy code):

1. *Main theorem chain* (Lemmas 2.2 and 2.3 and Proposition 2.5): pointer commutation, block-diagonal decomposition and exponential factorization, amplitude-independence of $P_k|\Psi(t)\rangle/a_k$ and derived record functionals, column-stochastic structure and non-negativity closure for $n = 2, 3, 4, 5$.

2. *Supporting results*: frozen diagonal (Corollary 3.1), basis covariance (Theorem 3.3) under generic SO(2) and phase rotations.

3. *Trajectory comparison* (Proposition 4.1 and Remark 4.3): martingale drift cancellation (binary and general-d), simplex preservation, absorbing-boundary vanishing.

4. *Robustness* (Proposition 3.7): general Lipschitz bound verified for $d_S = 2$ (2300 random trials), $d_S = 3$ (200 random trials), and non-rank-1 projectors (3000 random trials), with 0 violations. The bound subsumes constants previously obtained in four explicit $2 \times 2 \times 2$ models. In the tested regimes, the worst-case ratio actual/bound approaches 1 as $\delta \to 0$; this is empirical support only and does not strengthen the analytic theorem.

Limitations

The symbolic checks operate on representative finite-dimensional models, not on the abstract theorem statements directly. They confirm that the algebraic identities hold exactly in the scoped models and that the structural patterns (amplitude-independent quotients, drift cancellation, non-negativity closure) are present as claimed. They do not constitute a proof of the general theorems, which are established by the analytic arguments in the main text. The SymPy source code is included as a supplementary file in the Zenodo record accompanying this paper.